

ASYMPTOTIC LINEAR BOUNDS OF CASTELNUOVO-MUMFORD REGULARITY IN MULTIGRADED MODULES

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ABSTRACT. Let A be a Noetherian standard \mathbb{N} -graded algebra over an Artinian local ring A_0 . Let I_1, \dots, I_t be homogeneous ideals of A and M a finitely generated \mathbb{N} -graded A -module. We prove that there exist two integers k, k' such that

$$\operatorname{reg}(I_1^{n_1} \cdots I_t^{n_t} M) \leq (n_1 + \cdots + n_t)k + k' \quad \text{for all } n_1, \dots, n_t \in \mathbb{N}.$$

1. INTRODUCTION

Let I be a homogeneous ideal of a polynomial ring $S = K[X_1, \dots, X_d]$ over a field K with usual grading. Bertram, Ein and Lazarsfeld [1] have initiated the study of the Castelnuovo-Mumford regularity of I^n as a function of n by proving that if I is the defining ideal of a smooth complex projective variety, then $\operatorname{reg}(I^n)$ is bounded by a linear function of n .

Thereafter, Chandler [2] and Geramita, Gimigliano and Pitteloud [4] proved that if $\dim(S/I) \leq 1$, then $\operatorname{reg}(I^n) \leq n \cdot \operatorname{reg}(I)$ for all $n \geq 1$. This result does not hold true for higher dimension, due to an example of Sturmfels [6]. However, in [7, Theorem 3.6], Swanson proved that $\operatorname{reg}(I^n) \leq kn$ for all $n \geq 1$, where k is some integer.

Later, Cutkosky, Herzog and Trung [3], and Kodiyalam [5] independently proved that $\operatorname{reg}(I^n)$ can be expressed as a linear function of n for all sufficiently large n . Recently, Trung and Wang proved the above result in a more general way [8, Theorem 3.2]; if S is a standard graded ring over a commutative Noetherian ring with unity, I a homogeneous ideal of S and M a finitely generated graded S -module, then $\operatorname{reg}(I^n M)$ is asymptotically a linear function of n .

In this context, the natural question arises “what happens when we consider several ideals instead of just considering one ideal?”. More precisely, if I_1, \dots, I_t are homogeneous ideals of S and M is a finitely generated graded S -module, then what will be the behaviour of $\operatorname{reg}(I_1^{n_1} \cdots I_t^{n_t} M)$ as a function of (n_1, \dots, n_t) ?

Let $A = A_0[x_1, \dots, x_d]$ be a Noetherian standard \mathbb{N} -graded algebra over an Artinian local ring (A_0, \mathfrak{m}) . In particular, A can be a coordinate ring of any projective variety over any field with usual grading. Let I_1, \dots, I_t be homogeneous ideals of A and M a finitely generated \mathbb{N} -graded A -module. In this article, we prove that there exist two integers k, k' such that

$$(\dagger) \quad \operatorname{reg}(I_1^{n_1} \cdots I_t^{n_t} M) \leq (n_1 + \cdots + n_t)k + k' \quad \text{for all } n_1, \dots, n_t \in \mathbb{N}.$$

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The rest of the paper is organized as follows. We start by introducing some notations and terminologies in Section 2. In Section 3, we give some preliminaries on Castelnuovo-Mumford regularity and multigraded modules which we use in order to prove our main result. Finally, in Section 4, we prove (†) in several steps.

2. NOTATION

Throughout this article, \mathbb{N} denotes the set of all non-negative integers and t is any fixed positive integer. We use small letters with underline (e.g., \underline{n}) to denote elements of \mathbb{N}^t , and we use subscripts mainly to denote the coordinates of such an element, e.g., $\underline{n} = (n_1, n_2, \dots, n_t)$. In particular, for each $1 \leq i \leq t$, \underline{e}^i denotes the i^{th} standard basis element of \mathbb{N}^t . We denote $\underline{0}$ the element of \mathbb{N}^t with all components 0. Throughout, we use the partial order on \mathbb{N}^t defined by $\underline{n} \geq \underline{m}$ if and only if $n_i \geq m_i$ for all $1 \leq i \leq t$. Set $|\underline{n}| = n_1 + \dots + n_t$.

If R is an \mathbb{N}^t -graded ring and L is an \mathbb{N}^t -graded R -module, then by $L_{\underline{n}}$, we always mean the $\underline{n}^{\text{th}}$ graded component of L . By standard multigraded ring, we mean a multigraded ring which is generated in total degree one, i.e., R is a standard \mathbb{N}^t -graded ring if $R = R_{\underline{0}}[R_{\underline{e}^1}, \dots, R_{\underline{e}^t}]$. All rings, graded or not, are assumed commutative with identity.

3. PRELIMINARIES

Let $A = A_0[x_1, \dots, x_d]$ be a Noetherian standard \mathbb{N} -graded ring. Let A_+ be the ideal $\langle x_1, \dots, x_d \rangle$ of A generated by the elements of positive degree. Let M be a finitely generated \mathbb{N} -graded A -module. For every integer $i \geq 0$, we denote the i^{th} local cohomology module of M with respect to A_+ by $H_{A_+}^i(M)$. For every integer $i \geq 0$, we set

$$a_i(M) := \max \left\{ \mu : H_{A_+}^i(M)_\mu \neq 0 \right\}$$

if $H_{A_+}^i(M) \neq 0$ and $a_i(M) := -\infty$ otherwise. The Castelnuovo-Mumford regularity of M is defined by

$$\text{reg}(M) := \max \{ a_i(M) + i : i \geq 0 \}.$$

For a given short exact sequence of graded modules, by considering the corresponding long exact sequence of local cohomology modules, we can prove the following well-known result.

Lemma 3.1. *Let A be as above. If $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is a short exact sequence of finitely generated \mathbb{N} -graded A -modules, then we have the following.*

- (i) $\text{reg}(M_1) \leq \max\{\text{reg}(M_2), \text{reg}(M_3) + 1\}$.
- (ii) $\text{reg}(M_2) \leq \max\{\text{reg}(M_1), \text{reg}(M_3)\}$.
- (iii) $\text{reg}(M_3) \leq \max\{\text{reg}(M_1) - 1, \text{reg}(M_2)\}$.

We use the following well-known lemma to prove our main result inductively.

Lemma 3.2. *Let A be a Noetherian standard \mathbb{N} -graded ring and M a finitely generated \mathbb{N} -graded A -module. Let x be a homogeneous element in A of positive degree l . Then we have the following inequality:*

$$\text{reg}(M) \leq \max\{\text{reg}(0 :_M x), \text{reg}(M/xM) - l + 1\}.$$

Over polynomial rings over fields, if x is such that $\dim(0 :_M x) \leq 1$, then the inequality could be replaced by equality.

Now we give some preliminaries on \mathbb{N}^t -graded modules. We start with the following lemma.

Lemma 3.3. *Let $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{\underline{n}}$ be a Noetherian \mathbb{N}^t -graded ring, and let $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{\underline{n}}$ be a finitely generated \mathbb{N}^t -graded R -module. Set $A = R_{\underline{0}}$. Let J be an ideal of A . Then there exists a positive integer k such that*

$$J^m L_{\underline{n}} \cap H_J^0(L_{\underline{n}}) = 0 \quad \text{for all } \underline{n} \in \mathbb{N}^t \text{ and } m \geq k.$$

Proof. Let $I = JR$ be the ideal of R generated by J . Since R is Noetherian and L is a finitely generated R -module, then by Artin-Rees lemma, there exists a positive integer c such that

$$(3.1) \quad \begin{aligned} (I^m L) \cap H_I^0(L) &= I^{m-c} ((I^c L) \cap H_I^0(L)) \quad \text{for all } m \geq c \\ &\subseteq I^{m-c} H_I^0(L) \quad \text{for all } m \geq c. \end{aligned}$$

Now consider the ascending chain of submodules of L :

$$(0 :_L I) \subseteq (0 :_L I^2) \subseteq (0 :_L I^3) \subseteq \cdots.$$

Since L is a Noetherian R -module, there exists some l such that

$$(3.2) \quad (0 :_L I^l) = (0 :_L I^{l+1}) = (0 :_L I^{l+2}) = \cdots = H_I^0(L).$$

Set $k := c + l$. Then from (3.1) and (3.2), we have

$$(I^m L) \cap H_I^0(L) \subseteq I^{m-c} (0 :_L I^{l+1}) = 0 \quad \text{for all } m \geq k,$$

which gives $(J^m L_{\underline{n}}) \cap H_J^0(L_{\underline{n}}) = 0$ for all $\underline{n} \in \mathbb{N}^t$ and $m \geq k$. \square

Now we are aiming to obtain some invariant of multigraded module with the help of the following result.

Lemma 3.4. *Let R be a Noetherian standard \mathbb{N}^t -graded ring and L an \mathbb{N}^t -graded R -module finitely generated in degrees $\leq \underline{u}$. Set $A = R_{\underline{0}}$. Then we have the following.*

- (i) *For any $\underline{v} \geq \underline{u}$, $\text{ann}_A(L_{\underline{v}}) \subseteq \text{ann}_A(L_{\underline{n}})$ for all $\underline{n} \geq \underline{v}$, and hence $\dim_A(L_{\underline{v}}) \geq \dim_A(L_{\underline{n}})$ for all $\underline{n} \geq \underline{v}$.*
- (ii) *There exists $\underline{v} \in \mathbb{N}^t$ such that $\text{ann}_A(L_{\underline{n}}) = \text{ann}_A(L_{\underline{v}})$ for all $\underline{n} \geq \underline{v}$, and hence $\dim_A(L_{\underline{n}}) = \dim_A(L_{\underline{v}})$ for all $\underline{n} \geq \underline{v}$.*

Proof. (i) Let $\underline{v} \geq \underline{u}$. Since R is standard and L is an \mathbb{N}^t -graded R -module finitely generated in degrees $\leq \underline{u}$, for any $\underline{n} \geq \underline{v}$ ($\geq \underline{u}$), we have

$$L_{\underline{n}} = R_{\underline{e}_1}^{n_1 - v_1} R_{\underline{e}_2}^{n_2 - v_2} \cdots R_{\underline{e}_t}^{n_t - v_t} L_{\underline{v}},$$

which gives $\text{ann}_A(L_{\underline{v}}) \subseteq \text{ann}_A(L_{\underline{n}})$, and hence $\dim_A(L_{\underline{v}}) \geq \dim_A(L_{\underline{n}})$ for all $\underline{n} \geq \underline{v}$.

(ii) Consider $\mathcal{C} := \{\text{ann}_A(L_{\underline{n}}) : \underline{n} \geq \underline{u}\}$, a collection of ideals of A . Since A is Noetherian, \mathcal{C} has a maximal element $\text{ann}_A(L_{\underline{v}})$, say. Then by part (i), it follows that $\text{ann}_A(L_{\underline{n}}) = \text{ann}_A(L_{\underline{v}})$ for all $\underline{n} \geq \underline{v}$, and hence $\dim_A(L_{\underline{n}}) = \dim_A(L_{\underline{v}})$ for all $\underline{n} \geq \underline{v}$. \square

Let us introduce the following invariant of multigraded module on which we apply induction to prove our main result.

Definition 3.5. Let R be a Noetherian standard \mathbb{N}^t -graded ring and L a finitely generated \mathbb{N}^t -graded R -module. We call $\underline{v} \in \mathbb{N}^t$ an *annihilator stable point* of L if

$$\text{ann}_{R_{\underline{0}}}(L_{\underline{n}}) = \text{ann}_{R_{\underline{0}}}(L_{\underline{v}}) \quad \text{for all } \underline{n} \geq \underline{v}.$$

In this case, we call $s := \dim_{R_{\underline{0}}}(L_{\underline{v}})$ as the *saturated dimension* of L .

Remark 3.6. Existence of an annihilator stable point of L (with the hypothesis given in the Definition 3.5) follows from Lemma 3.4(ii). Let $\underline{v}, \underline{w} \in \mathbb{N}^t$ be two annihilator stable points of L , i.e.,

$$\begin{aligned} \text{ann}_{R_0}(L_{\underline{n}}) &= \text{ann}_{R_0}(L_{\underline{v}}) \quad \text{for all } \underline{n} \geq \underline{v} \\ \text{and } \text{ann}_{R_0}(L_{\underline{n}}) &= \text{ann}_{R_0}(L_{\underline{w}}) \quad \text{for all } \underline{n} \geq \underline{w}. \end{aligned}$$

If we denote $\dim_{R_0}(L_{\underline{v}})$ and $\dim_{R_0}(L_{\underline{w}})$ by $s(\underline{v})$ and $s(\underline{w})$ respectively, then observe that $s(\underline{v}) = s(\underline{w})$. Thus the saturated dimension of L is well-defined.

Let us recall the following result from [9, Lemma 3.3].

Lemma 3.7. *Let R be a Noetherian standard \mathbb{N}^t -graded ring and L a finitely generated \mathbb{N}^t -graded R -module. For any fixed integers $1 \leq i \leq t$ and $\lambda \in \mathbb{N}$, set*

$$S_i := \bigoplus_{\{\underline{n} \in \mathbb{N}^t : n_i = 0\}} R_{\underline{n}} \quad \text{and} \quad M_{i\lambda} := \bigoplus_{\{\underline{n} \in \mathbb{N}^t : n_i = \lambda\}} L_{\underline{n}}.$$

Then S_i is a Noetherian standard \mathbb{N}^{t-1} -graded ring and $M_{i\lambda}$ is a finitely generated \mathbb{N}^{t-1} -graded S_i -module.

Discussion 3.8. Let

$$R = \bigoplus_{(\underline{n}, i) \in \mathbb{N}^{t+1}} R_{(\underline{n}, i)} \quad \text{and} \quad L = \bigoplus_{(\underline{n}, i) \in \mathbb{N}^{t+1}} L_{(\underline{n}, i)}$$

be a Noetherian \mathbb{N}^{t+1} -graded ring and a finitely generated \mathbb{N}^{t+1} -graded R -module respectively. For each $\underline{n} \in \mathbb{N}^t$, we set

$$R_{(\underline{n}, \star)} := \bigoplus_{i \in \mathbb{N}} R_{(\underline{n}, i)} \quad \text{and} \quad L_{(\underline{n}, \star)} := \bigoplus_{i \in \mathbb{N}} L_{(\underline{n}, i)}.$$

We give \mathbb{N}^t -grading structures on

$$R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n}, \star)} \quad \text{and} \quad L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{(\underline{n}, \star)}$$

in the obvious way, i.e., by setting $R_{(\underline{n}, \star)}$ and $L_{(\underline{n}, \star)}$ as the $\underline{n}^{\text{th}}$ graded components of R and L respectively. Then clearly, for any $\underline{m}, \underline{n} \in \mathbb{N}^t$, we have

$$R_{(\underline{m}, \star)} \cdot R_{(\underline{n}, \star)} \subseteq R_{(\underline{m}+\underline{n}, \star)} \quad \text{and} \quad R_{(\underline{m}, \star)} \cdot L_{(\underline{n}, \star)} \subseteq L_{(\underline{m}+\underline{n}, \star)}.$$

Thus R is an \mathbb{N}^t -graded ring and L is an \mathbb{N}^t -graded R -module. Since we are changing only the grading, R is anyway Noetherian. Since L is finitely generated \mathbb{N}^{t+1} -graded R -module, it is just an observation that $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{(\underline{n}, \star)}$ is finitely generated as \mathbb{N}^t -graded R -module. Now we set $A := R_{(\underline{0}, \star)}$. Note that A is a Noetherian \mathbb{N} -graded ring, and for each $\underline{n} \in \mathbb{N}^t$, $R_{(\underline{n}, \star)}$ and $L_{(\underline{n}, \star)}$ are finitely generated \mathbb{N} -graded A -modules.

We are going to refer the following hypothesis repeatedly in the rest of the paper.

Hypothesis 3.9. Let

$$R = \bigoplus_{(\underline{n}, i) \in \mathbb{N}^{t+1}} R_{(\underline{n}, i)}$$

be a Noetherian \mathbb{N}^{t+1} -graded ring, *which need not be standard*. Let

$$L = \bigoplus_{(\underline{n}, i) \in \mathbb{N}^{t+1}} L_{(\underline{n}, i)}$$

be a finitely generated \mathbb{N}^{t+1} -graded R -module. For each $\underline{n} \in \mathbb{N}^t$, we set

$$R_{(\underline{n}, \star)} := \bigoplus_{i \in \mathbb{N}} R_{(\underline{n}, i)} \quad \text{and} \quad L_{(\underline{n}, \star)} := \bigoplus_{i \in \mathbb{N}} L_{(\underline{n}, i)}.$$

Also set $A := R_{(\underline{0}, \star)}$. Suppose $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n}, \star)}$ and $A = R_{(\underline{0}, \star)}$ are standard as \mathbb{N}^t -graded ring and \mathbb{N} -graded ring respectively, i.e.

$$R = R_{(\underline{0}, \star)}[R_{(\underline{e}^1, \star)}, R_{(\underline{e}^2, \star)}, \dots, R_{(\underline{e}^t, \star)}] \quad \text{and} \quad R_{(\underline{0}, \star)} = R_{(\underline{0}, 0)}[R_{(\underline{0}, 1)}].$$

Assume $A_0 = R_{(\underline{0}, 0)}$ is Artinian local with the maximal ideal \mathfrak{m} . Since A is a Noetherian standard \mathbb{N} -graded ring, we assume $A = A_0[x_1, \dots, x_d]$ for some $x_1, \dots, x_d \in A_1$. Let $A_+ = \langle x_1, \dots, x_d \rangle$.

With the Hypothesis 3.9, from Discussion 3.8, we have the following.

- (0) $R = \bigoplus_{(\underline{n}, i) \in \mathbb{N}^{t+1}} R_{(\underline{n}, i)}$ is not necessarily standard as \mathbb{N}^{t+1} -graded ring.
- (1) $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n}, \star)}$ is a Noetherian standard \mathbb{N}^t -graded ring.
- (2) $A = R_{(\underline{0}, \star)}$ is a Noetherian standard \mathbb{N} -graded ring.
- (3) $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{(\underline{n}, \star)}$ is a finitely generated \mathbb{N}^t -graded R -module.
- (4) For each $\underline{n} \in \mathbb{N}^t$, $R_{(\underline{n}, \star)}$ and $L_{(\underline{n}, \star)}$ are finitely generated \mathbb{N} -graded A -modules.

Here is an example which satisfies the Hypothesis 3.9.

Example 3.10. Let A be a Noetherian standard \mathbb{N} -graded algebra over an Artinian local ring A_0 . Let I_1, \dots, I_t be homogeneous ideals of A and M a finitely generated \mathbb{N} -graded A -module. Let $R = A[I_1 T_1, \dots, I_t T_t]$ be the Rees algebra of I_1, \dots, I_t over the graded ring A and let $L = M[I_1 T_1, \dots, I_t T_t]$ be the Rees module of M with respect to the ideals I_1, \dots, I_t . We give \mathbb{N}^{t+1} -grading structures on R and L by setting $(\underline{n}, i)^{\text{th}}$ graded components of R and L as the i^{th} graded components of the \mathbb{N} -graded A -modules $I_1^{n_1} \dots I_t^{n_t} A$ and $I_1^{n_1} \dots I_t^{n_t} M$ respectively. Then clearly, R is a Noetherian \mathbb{N}^{t+1} -graded ring and L is a finitely generated \mathbb{N}^{t+1} -graded R -module. Note that R is not necessarily standard as \mathbb{N}^{t+1} -graded ring. Also note that for each $\underline{n} \in \mathbb{N}^t$,

$$R_{(\underline{n}, \star)} = \bigoplus_{i \in \mathbb{N}} R_{(\underline{n}, i)} = I_1^{n_1} \dots I_t^{n_t} A$$

$$\text{and} \quad L_{(\underline{n}, \star)} = \bigoplus_{i \in \mathbb{N}} L_{(\underline{n}, i)} = I_1^{n_1} \dots I_t^{n_t} M.$$

Since $R = A[I_1 T_1, \dots, I_t T_t] = R_{(\underline{0}, \star)}[R_{(\underline{e}^1, \star)}, \dots, R_{(\underline{e}^t, \star)}]$, $R = \bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n}, \star)}$ is standard as \mathbb{N}^t -graded ring. Thus R and L are satisfying the Hypothesis 3.9.

From now onwards, by R and L , we mean \mathbb{N}^t -graded ring $\bigoplus_{\underline{n} \in \mathbb{N}^t} R_{(\underline{n}, \star)}$ and \mathbb{N}^t -graded R -module $\bigoplus_{\underline{n} \in \mathbb{N}^t} L_{(\underline{n}, \star)}$ (satisfying the Hypothesis 3.9) respectively.

4. LINEAR BOUNDS OF REGULARITY

In this section, we are aiming to prove that the regularity of $L_{(\underline{n}, \star)}$ as an \mathbb{N} -graded A -module is bounded by a linear function of \underline{n} by using induction on the saturated dimension of the \mathbb{N}^t -graded R -module L . Here is the base case.

Theorem 4.1. *With the Hypothesis 3.9, let $L = \bigoplus_{\underline{n} \in \mathbb{N}^t} L_{(\underline{n}, \star)}$ be generated in degrees $\leq \underline{u}$. If $\dim_A(L_{(\underline{v}, \star)}) = 0$ for some $\underline{v} \geq \underline{u}$, then there exists an integer k such that*

$$\operatorname{reg}(L_{(\underline{n}, \star)}) < |\underline{n} - \underline{u}|k + k \quad \text{for all } \underline{n} \geq \underline{v}.$$

Proof. Let $\dim_A(L_{(\underline{v}, \star)}) = 0$ for some $\underline{v} \geq \underline{u}$. Then from Lemma 3.4(i), we have

$$\dim_A(L_{(\underline{n}, \star)}) = 0 \quad \text{for all } \underline{n} \geq \underline{v}.$$

By Grothendieck vanishing theorem, we have

$$H_{A_+}^i(L_{(\underline{n}, \star)}) = 0 \quad \text{for all } i > 0 \text{ and } \underline{n} \geq \underline{v}.$$

Therefore in this case

$$(4.1) \quad \operatorname{reg}(L_{(\underline{n}, \star)}) = \max \left\{ \mu : H_{A_+}^0(L_{(\underline{n}, \star)})_\mu \neq 0 \right\} \quad \text{for all } \underline{n} \geq \underline{v}.$$

Now consider the finite collection

$$\mathcal{D} := \{R_{(\underline{e}^1, \star)}, R_{(\underline{e}^2, \star)}, \dots, R_{(\underline{e}^t, \star)}, L_{(\underline{u}, \star)}\}.$$

Since each member of \mathcal{D} is finitely generated \mathbb{N} -graded $A = A_0[x_1, \dots, x_d]$ -module, we may assume that every member of \mathcal{D} is generated in degrees $\leq k_1$ for some $k_1 \in \mathbb{N}$. Since L is a finitely generated \mathbb{N}^t -graded R -module and A_+ is an ideal of $A (= R_{(\underline{0}, \star)})$, by Lemma 3.3, there exists a positive integer k_2 such that

$$(4.2) \quad (A_+)^{k_2} L_{(\underline{n}, \star)} \cap H_{A_+}^0(L_{(\underline{n}, \star)}) = 0 \quad \text{for all } \underline{n} \in \mathbb{N}^t.$$

Now set $k := k_1 + k_2$. We claim that

$$(4.3) \quad H_{A_+}^0(L_{(\underline{n}, \star)})_\mu = 0 \quad \text{for all } \underline{n} \geq \underline{v} \text{ and } \mu \geq |\underline{n} - \underline{u}|k + k.$$

To show (4.3), fix $\underline{n} \geq \underline{v}$ and $\mu \geq |\underline{n} - \underline{u}|k + k$. Assume $X \in H_{A_+}^0(L_{(\underline{n}, \star)})_\mu$. Note that the homogeneous (with respect to \mathbb{N} -grading over A) element X of

$$L_{(\underline{n}, \star)} = R_{(\underline{e}^1, \star)}^{n_1 - u_1} R_{(\underline{e}^2, \star)}^{n_2 - u_2} \cdots R_{(\underline{e}^t, \star)}^{n_t - u_t} L_{(\underline{u}, \star)}$$

can be written as a finite sum of elements of the following type:

$$(r_{11}r_{12} \cdots r_{1 \ n_1 - u_1})(r_{21}r_{22} \cdots r_{2 \ n_2 - u_2}) \cdots (r_{t1}r_{t2} \cdots r_{t \ n_t - u_t})Y$$

for some homogeneous (with respect to \mathbb{N} -grading over A) elements

$$r_{i1}, r_{i2}, \dots, r_{i \ n_i - u_i} \in R_{(\underline{e}^i, \star)} \quad \text{for all } 1 \leq i \leq t, \text{ and } Y \in L_{(\underline{u}, \star)}.$$

Considering the homogeneous degree with respect to \mathbb{N} -grading over A , we have

$$\deg(Y) + \sum_{i=1}^t \{\deg(r_{i1}) + \deg(r_{i2}) + \cdots + \deg(r_{i \ n_i - u_i})\} = \mu \geq |\underline{n} - \underline{u}|k + k,$$

which gives at least one of the elements

$$r_{11}, r_{12}, \dots, r_{1 \ n_1 - u_1}, \dots, r_{t1}, r_{t2}, \dots, r_{t \ n_t - u_t} \quad \text{and } Y$$

is of degree $\geq k$. In first case, we consider $\deg(r_{ij}) \geq k$ for some i, j . Since $R_{(\underline{e}^i, \star)}$ is an \mathbb{N} -graded A -module generated in degrees $\leq k_1$, we have

$$\begin{aligned} r_{ij} &\in (R_{(\underline{e}^i, \star)})_{\deg(r_{ij})} = (A_1)^{\deg(r_{ij}) - k_1} (R_{(\underline{e}^i, \star)})_{k_1} \\ &\subseteq (A_+)^{k_2} R_{(\underline{e}^i, \star)} \quad [\text{as } \deg(r_{ij}) - k_1 \geq k - k_1 = k_2]. \end{aligned}$$

In another case, we consider $\deg(Y) \geq k$. In this case also, since $L_{(\underline{u}, \star)}$ is an \mathbb{N} -graded A -module generated in degrees $\leq k_1$, we have

$$\begin{aligned} Y \in (L_{(\underline{u}, \star)})_{\deg(Y)} &= (A_1)^{\deg(Y)-k_1} (L_{(\underline{u}, \star)})_{k_1} \\ &\subseteq (A_+)^{k_2} L_{(\underline{u}, \star)} \quad [\text{as } \deg(Y) - k_1 \geq k - k_1 = k_2]. \end{aligned}$$

In both cases, the typical element $(r_{11}r_{12} \cdots r_{1 \ n_1-u_1}) \cdots (r_{t1}r_{t2} \cdots r_{t \ n_t-u_t})Y$ is in

$$(A_+)^{k_2} R_{(\underline{e}^1, \star)}^{n_1-u_1} R_{(\underline{e}^2, \star)}^{n_2-u_2} \cdots R_{(\underline{e}^t, \star)}^{n_t-u_t} L_{(\underline{u}, \star)} = (A_+)^{k_2} L_{(\underline{n}, \star)},$$

and hence $X \in (A_+)^{k_2} L_{(\underline{n}, \star)}$. Therefore $X \in (A_+)^{k_2} L_{(\underline{n}, \star)} \cap H_{A_+}^0(L_{(\underline{n}, \star)})$, which gives $X = 0$ by (4.2). Thus we have

$$H_{A_+}^0(L_{(\underline{n}, \star)})_\mu = 0 \quad \text{for all } \underline{n} \geq \underline{v} \text{ and } \mu \geq |\underline{n} - \underline{v}|k + k,$$

and hence the theorem follows from (4.1). \square

Now we give the inductive step to prove the following linear boundedness result.

Theorem 4.2. *With the Hypothesis 3.9, there exist $\underline{u} \in \mathbb{N}^t$ and an integer k such that*

$$\text{reg}(L_{(\underline{n}, \star)}) < |\underline{n}|k + k \quad \text{for all } \underline{n} \geq \underline{u}.$$

In particular, if $t = 1$, then there exist two integers k, k' such that

$$\text{reg}(L_{(n, \star)}) \leq nk + k' \quad \text{for all } n \in \mathbb{N}.$$

Proof. Let $\underline{v} \in \mathbb{N}^t$ be an annihilator stable point of L and s the saturated dimension of L . Without loss of generality, we may assume that L is finitely generated as R -module in degrees $\leq \underline{v}$. We prove the theorem by induction on s . If $s = 0$, then the theorem follows from Theorem 4.1 by taking $\underline{u} := \underline{v}$. Therefore we may as well assume that $s > 0$ and the theorem holds true for all such finitely generated \mathbb{N}^t -graded R -modules with the saturated dimensions $\leq s - 1$.

Let $\mathfrak{n} := \mathfrak{m} \oplus A_+$ be the maximal homogeneous ideal of A . We claim that

$$\mathfrak{n} \notin \text{Min}(A/\text{ann}_A(L_{(\underline{v}, \star)})).$$

Since the collection of all minimal prime ideals of A containing $\text{ann}_A(L_{(\underline{v}, \star)})$ are associated prime ideals of $A/\text{ann}_A(L_{(\underline{v}, \star)})$, they are homogeneous, and hence they must be contained in \mathfrak{n} . Thus if the above claim is not true, then we have

$$\text{Min}(A/\text{ann}_A(L_{(\underline{v}, \star)})) = \{\mathfrak{n}\},$$

and hence $s = \dim_A(L_{(\underline{v}, \star)}) = 0$, which is a contradiction. Therefore the above claim is true, and hence by prime avoidance lemma and using the fact that \mathfrak{n} is the only homogeneous prime ideal of A containing A_+ (as (A_0, \mathfrak{m}) is Artinian local), we have

$$A_+ \not\subseteq \bigcup \{P : P \in \text{Min}(A/\text{ann}_A(L_{(\underline{v}, \star)}))\}.$$

Then by graded version of prime avoidance lemma, we may choose a homogeneous element x in A of positive degree such that

$$x \notin \bigcup \{P : P \in \text{Min}(A/\text{ann}_A(L_{(\underline{v}, \star)}))\}.$$

Note that $\text{ann}_A(L_{(\underline{n}, \star)}) = \text{ann}_A(L_{(\underline{v}, \star)})$ for all $\underline{n} \geq \underline{v}$. Therefore for all $\underline{n} \geq \underline{v}$, we have

$$\begin{aligned} \dim_A(L_{(\underline{n}, \star)}/xL_{(\underline{n}, \star)}), \dim_A(0 :_{L_{(\underline{n}, \star)}} x) &\leq \dim_A(L_{(\underline{n}, \star)}) - 1 = s - 1 \\ \text{as } \text{ann}_A(L_{(\underline{n}, \star)}/xL_{(\underline{n}, \star)}), \text{ann}_A(0 :_{L_{(\underline{n}, \star)}} x) &\supseteq \langle \text{ann}_A(L_{(\underline{n}, \star)}), x \rangle. \end{aligned}$$

Now observe that L/xL and $(0 :_L x)$ are finitely generated \mathbb{N}^t -graded R -modules with saturated dimensions $\leq s-1$. Therefore, by induction hypothesis, there exist \underline{w} and \underline{w}' in \mathbb{N}^t and two integers k_1, k_2 such that

$$\begin{aligned} \operatorname{reg}(L_{(\underline{n}, \star)}/xL_{(\underline{n}, \star)}) &< |\underline{n}|k_1 + k_1 \quad \text{for all } \underline{n} \geq \underline{w} \\ \text{and } \operatorname{reg}(0 :_{L_{(\underline{n}, \star)}} x) &< |\underline{n}|k_2 + k_2 \quad \text{for all } \underline{n} \geq \underline{w}'. \end{aligned}$$

Set $k := \max\{k_1, k_2\}$ and $\underline{u} := \max\{\underline{w}, \underline{w}'\}$ (i.e., $u_i := \max\{w_i, w'_i\}$ for all $1 \leq i \leq t$, and $\underline{u} := (u_1, \dots, u_t)$). Then from Lemma 3.2, we have

$$\begin{aligned} \operatorname{reg}(L_{(\underline{n}, \star)}) &\leq \max\{\operatorname{reg}(L_{(\underline{n}, \star)}/xL_{(\underline{n}, \star)}), \operatorname{reg}(0 :_{L_{(\underline{n}, \star)}} x)\} \\ &< |\underline{n}|k + k \quad \text{for all } \underline{n} \geq \underline{u}. \end{aligned}$$

This completes the proof of the first part of the theorem. To prove the second part, assume $t = 1$. Then from the first part, there exist $u \in \mathbb{N}$ and an integer k such that

$$\operatorname{reg}(L_{(n, \star)}) < nk + k \quad \text{for all } n \geq u.$$

Set $k' := \max\{k, \operatorname{reg}(L_{(n, \star)}) : 0 \leq n \leq u-1\}$. Then clearly, we have

$$\operatorname{reg}(L_{(n, \star)}) \leq nk + k' \quad \text{for all } n \in \mathbb{N},$$

which completes the proof of the theorem. \square

Above theorem gives the result that $\operatorname{reg}(L_{(\underline{n}, \star)})$ has linear bound for all $\underline{n} \geq \underline{u}$, for some $\underline{u} \in \mathbb{N}^t$. Now we prove the result for all $\underline{n} \in \mathbb{N}^t$.

Theorem 4.3. *With the Hypothesis 3.9, there exist two integers k, k' such that*

$$\operatorname{reg}(L_{(\underline{n}, \star)}) \leq (n_1 + \dots + n_t)k + k' \quad \text{for all } \underline{n} \in \mathbb{N}^t.$$

Proof. We prove the theorem by induction on t . If $t = 1$, then the theorem follows from the second part of the Theorem 4.2. Therefore we may as well assume that $t \geq 2$ and the theorem holds true for $t-1$.

By Theorem 4.2, there exist $\underline{u} \in \mathbb{N}^t$ and an integer k_1 such that

$$(4.4) \quad \operatorname{reg}(L_{(\underline{n}, \star)}) < |\underline{n}|k_1 + k_1 \quad \text{for all } \underline{n} \geq \underline{u}.$$

Now for each $1 \leq i \leq t$ and $0 \leq \lambda < u_i$, we set

$$S_i := \bigoplus_{\{\underline{n} \in \mathbb{N}^t : n_i = 0\}} R_{(\underline{n}, \star)} \quad \text{and} \quad M_{i\lambda} := \bigoplus_{\{\underline{n} \in \mathbb{N}^t : n_i = \lambda\}} L_{(\underline{n}, \star)}.$$

Then from Lemma 3.7, S_i is a Noetherian standard \mathbb{N}^{t-1} -graded ring and $M_{i\lambda}$ is a finitely generated \mathbb{N}^{t-1} -graded S_i -module. Therefore, by induction hypothesis, for each $1 \leq i \leq t$ and $0 \leq \lambda < u_i$, there exist two integers $k_{i\lambda}$ and $k'_{i\lambda}$ such that

$$\begin{aligned} \operatorname{reg}(L_{(n_1, \dots, n_{i-1}, \lambda, n_{i+1}, \dots, n_t, \star)}) &\leq (n_1 + \dots + n_{i-1} + n_{i+1} + \dots + n_t)k_{i\lambda} + k'_{i\lambda} \\ (4.5) \quad &= (n_1 + \dots + n_{i-1} + \lambda + n_{i+1} + \dots + n_t)k_{i\lambda} + k''_{i\lambda} \\ &\quad \text{for all } n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_t \in \mathbb{N}, \end{aligned}$$

where $k''_{i\lambda} = k'_{i\lambda} - \lambda k_{i\lambda}$. Now set

$$\begin{aligned} k &:= \max\{k_1, k_{i\lambda} : 1 \leq i \leq t, 0 \leq \lambda < u_i\} \quad \text{and} \\ k' &:= \max\{k_1, k''_{i\lambda} : 1 \leq i \leq t, 0 \leq \lambda < u_i\}. \end{aligned}$$

We claim that

$$(4.6) \quad \operatorname{reg}(L_{(\underline{n}, \star)}) \leq |\underline{n}|k + k' \quad \text{for all } \underline{n} \in \mathbb{N}^t.$$

To prove (4.6), consider an arbitrary $\underline{n} \in \mathbb{N}^t$. If $\underline{n} \geq \underline{u}$, then (4.6) follows from (4.4). Otherwise if $\underline{n} \not\geq \underline{u}$, then we have $n_i < u_i$ for at least one $i \in \{1, \dots, t\}$, and hence in this case, (4.6) holds true by (4.5). \square

Now we have arrived at the main goal of this article.

Corollary 4.4. *Let A be a Noetherian standard \mathbb{N} -graded algebra over an Artinian local ring A_0 . Let I_1, \dots, I_t be homogeneous ideals of A and M a finitely generated \mathbb{N} -graded A -module. Then there exist two integers k, k' such that*

$$\text{reg}(I_1^{n_1} \cdots I_t^{n_t} M) \leq (n_1 + \cdots + n_t)k + k' \quad \text{for all } n_1, \dots, n_t \in \mathbb{N}.$$

Proof. Let $R = A[I_1 T_1, \dots, I_t T_t]$ be the Rees algebra of I_1, \dots, I_t over the graded ring A and let $L = M[I_1 T_1, \dots, I_t T_t]$ be the Rees module of M with respect to the ideals I_1, \dots, I_t . We give \mathbb{N}^{t+1} -grading structures on R and L by setting $(\underline{n}, i)^{\text{th}}$ graded components of R and L as the i^{th} graded components of the \mathbb{N} -graded A -modules $I_1^{n_1} \cdots I_t^{n_t} A$ and $I_1^{n_1} \cdots I_t^{n_t} M$ respectively. From Example 3.10, note that R and L are satisfying the Hypothesis 3.9, and in this case

$$L_{(\underline{n}, \star)} = I_1^{n_1} \cdots I_t^{n_t} M \quad \text{for all } \underline{n} \in \mathbb{N}^t.$$

Therefore the corollary follows from Theorem 4.3. \square

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